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AN APPROXIMATION TO $\Omega^n\Sigma^n X$

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ABSTRACT. For an arbitrary (nonconnected) based space X, a geometrical construction $\tilde{C}_n X$ is given, such that $\tilde{C}_n X$ is weakly homotopy-equivalent to $\Omega^n\Sigma^n X$ as a $\mathcal{C}_n$-space.

In *The geometry of iterated loop spaces* [3], J. P. May constructs functors C_n from spaces to $\mathcal{C}_n$-spaces and natural maps α_n: $C_n X \to \Omega^n\Sigma^n X$ which are weak homotopy equivalences if X is connected. We present here a related construction giving a functor $\tilde{C}_n X$ which is an approximation to $\Omega^n\Sigma^n X$ for arbitrary X.

The philosophy behind our approach is that where May uses subcubes of the unit n-cube I^n in his construction, we use signed subcubes and allow them to merge along a single coordinate; these are to account for the homotopy inverses that one loses when X is not connected. So far as we know, the first person to attempt to exploit this philosophy was Dusa McDuff [4]; surprisingly, the resulting space for particles has the wrong homotopy type, as she shows there. Apparently, a space of configurations of points is just too crude to give a good approximation.

In the present work, we construct a natural map α_n: $\tilde{C}_n X \to \Omega^n\Sigma^n X$. If α_{n-1}: $C_{n-1}\Sigma X \to \Omega^{n-1}\Sigma^n X$ is May's approximation, α_n factors as a composite

$$\tilde{C}_n X \xrightarrow{\beta_n} \Omega C_{n-1}\Sigma X \xrightarrow{\Omega\alpha_{n-1}} \Omega^n\Sigma^n X;$$

it will then follow that α_n is an equivalence if β_n is, since ΣX is connected. To study β_n we introduce a space $\tilde{C}_n' X$ and a map β_n' from this space to the Moore loop space $\Lambda C_{n-1} X$, such that we have a commutative diagram

$$\begin{array}{ccc} \tilde{C}_n' X & \xrightarrow{\beta_n'} & \Lambda C_{n-1}\Sigma X \\ \mu\downarrow & & \downarrow\pi \\ \tilde{C}_n X & \xrightarrow{\beta_n} & \Omega C_{n-1}\Sigma X \end{array}$$

where μ and π are homotopy equivalences.

We then proceed by producing a quasifibration p: $E_n X \to C_{n-1}\Sigma X$ with quasifiber $\tilde{C}_n' X$ and contractible total space, together with a comparison

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$$\begin{array}{ccc}
\tilde{C}'_n X & \xrightarrow{\beta'_n} & \Lambda C_{n-1}\Sigma X \\
\downarrow\cap & & \downarrow\cap \\
E_n X & \to & \Pi C_{n-1}\Sigma X \\
p\downarrow & & \downarrow p_e \\
C_{n-1}\Sigma X & \to & C_{n-1}\Sigma X
\end{array}$$

of p to the Moore path space fibration p_e. It will thus follow that β'_n is an equivalence, and hence so are β_n and α_n.

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1. Basic definitions. A *little n-cube* is (as in [3]) a linear embedding $c\colon I^n \to I^n$ which is the product of n increasing linear functions $c_i\colon I \to I$.

In [3], the little n-cubes operad $\mathcal{C}_n$ is defined as a sequence of spaces $\mathcal{C}_{n,j}$, together with some structure maps relating them. Each $\mathcal{C}_{n,j}$ is the space of j-tuples of little n-cubes whose images intersect only on the image of the boundary of their domains; we will use the term *nonoverlapping* for this condition.

The operad $\mathcal{C}_n$ leads to a model for $\Omega^n\Sigma^n X$ in which each little n-cube represents a map from S^n to S^n of degree one. These are "labelled" by points of X, so that a point in this model $C_n X$ corresponds to a sequence of j maps $S^n \to \{x_i\}^+ \wedge S^n \hookrightarrow \Sigma^n X$, and one takes the "union" of these maps in order to obtain a map in $\Omega^n\Sigma^n X$. We are trying to examine the case in which X is not connected. In this case, one needs to allow maps of degree -1 to provide "homotopy inverses" to the maps of degree $+1$. Also, the intermediate stages between (for example) a configuration with two oppositely oriented cubes, and the empty configuration, must be represented. This leads to the concept of a "partial" little n-cube, which can be thought of as representing one piece of a piecewise linear map.

A *partial little n-cube* is defined to be an embedding $c\colon A \times I^{n-1} \to I^n$ which is the product of an affine embedding c' of a closed nonempty subinterval A of I into I, and a little $(n-1)$-cube c''. Note that c' may be increasing or decreasing (and we use the term *positive* or *negative*, respectively, to describe c), and A is allowed to be a single point (in which case we say c is *degenerate*).

We say that a partial little n-cube $c = c' \times c''$ *has slope m* ($m \neq 0$) if either A is a point or c' has the form $c'(t) = mt + b$ for some b, for $t \in A$; thus degenerate cubes have all nonzero slopes.

Here we wish to define a new space $\tilde{\mathcal{C}}_{n,j}$ using partial little n-cubes, but we need to have the cubes "fit together" properly. So we will say that two nonoverlapping cubes $c\colon A \times I^{n-1} \to I^n$ and $d\colon B \times I^{n-1} \to I^n$ are *attached at a* (where $a \in \partial A \cap \partial B \cap (0, 1)$) if $c'(a) = d'(a)$ and $c'' = d''$; that is, if c and d agree on $\{a\} \times I^{n-1}$. We say that c is *attached from below by d* (and, equivalently, d is *attached from above by c*) if $c'(s) > d'(t)$ whenever $s \in \operatorname{Int} A$ or $t \in \operatorname{Int} B$.

We say that a set $C = \{c_1, \dots, c_k\}$ of partial little n-cubes is an *admissible configuration if*

(i) the cubes in C are nonoverlapping, all have slope $m_r = \pm m$ for some fixed m, and

(ii) for all $c \in C$, $c\colon A \times I^{n-1} \to I^n$, if $a \in \partial A$ and a is different from 0 and 1, then c is attached at a by some nondegenerate cube $d \in C$.

We now define $\tilde{\mathcal{C}}_{n,j}$ to be the set of j-tuples $\langle c_1, \ldots, c_j\rangle$ of partial little n-cubes $c_r\colon A \times I^{n-1} \to I^n$ which are nonoverlapping and such that $\{c_1, \ldots, c_j\}$ is a disjoint union of admissible configurations. We make $\tilde{\mathcal{C}}_{n,j}$ into a space by identifying each cube $c\colon [a^1, a^2] \times I^{n-1} \to I^n$ with a tuple $(a^1, a^2, \tilde{c}', c'')$ in $I \times I \times \mathrm{Map}(I, I) \times \mathcal{C}_{n-1,1}$, where $\tilde{c}'$ is the piecewise linear function which equals c' on $[a^1, a^2]$ and is constant on $[0, a^1]$ and $[a^2, 1]$. Then we topologize $\tilde{\mathcal{C}}_{n,j}$ as a subspace of $(I \times I \times \mathrm{Map}(I, I) \times \mathcal{C}_{n-1,1})^j$ under these identifications.

The partial little cubes will occur only as parts of so-called "closed configurations" of little cubes, ones which have no unattached endpoint except on $\partial I \times I^{n-1}$, and such a configuration represents a map of degree $+1$, -1, or 0, with finitely many changes of direction in the first coordinate. Note that we allow two successive cubes of the same slope, and introduce degeneracies later on to make this equivalent to one larger cube. This is necessary to account for the possibility of "smoothing out a wrinkle"; that is, where three cubes are attached in a row with the middle cube of opposite sign from the other two. To allow the middle one to shrink and become degenerate, we must topologize the space so that the resulting configuration is equivalent to one with the two end cubes replaced with one larger one which is their union.

To facilitate this equivalence, we will define the union of two partial little cubes c and d, which are attached and have the same slope. Note that this implies that their domains $A \times I^{n-1}$ and $B \times I^{n-1}$ have disjoint interiors (since c and d are nonoverlapping), and agree as maps when restricted to the intersection of their domains at the boundary. Consequently, we define their *union* $c \cup d\colon (A \cup B) \times I^{n-1} \to I^n$ to be equal to c on $A \times I^{n-1}$ and to d on $B \times I^{n-1}$. It is trivial to check that $c \cup d$ is a linear embedding, and hence it is a partial little n-cube.

We introduce here a weaker notion than that of attachment. We will call c and d *adjacent* if they are attached or if they satisfy all the data for being attached at a point a except that $a = 0$ or 1, and have opposite orientation (i.e., are not both positive or negative).

We think of attachment as an "open condition" in that it holds for two cubes c_r and c_s throughout a whole neighborhood in $\tilde{\mathcal{C}}_{n,j}$, and adjacency is a "closed condition" in that "nonadjacency" holds throughout a neighborhood. In some sense we can think of adjacency as the "closure" of attachment.

Some further notation will make the technical arguments go more smoothly. Any partial little cube c_r has the general form $c_r = c_r' \times c_r''\colon [a_r^1, a_r^2] \times I^{n-1} \to I^n$, where

$$c_r(t_1, \ldots, t_n) = (m_{r,1}t_1 + b_1, \ldots, m_{r,n}t_n + b_n)$$

where $m_{r,1} \neq 0$ and $m_{r,i} > 0$ for $2 \leqslant i \leqslant n$. We shall refer to $m_{r,1}$ (or just m_r for short) as the *slope* of c_r, the other slopes not being of technical importance. In any such cube the endpoints a_r^1 and a_r^2 may be labelled by a_r^+ and a_r^- so that

$c_r'(a_r^+) \geqslant c_r'(a_r^-)$. We will use the notation a_r^ε where ε stands for $+$ or $-$, and $a_r^{-\varepsilon}$ will mean a_r^+ if $\varepsilon = -$, and a_r^- if $\varepsilon = +$.

In [5], Graeme Segal discusses a variant of C_nX which produces a strict topological monoid; this seems to be a model for the Moore loop space $\Lambda(\Omega^{n-1}\Sigma^nX)$. Recall that ΛX consists of points $(a; \omega)$ such that $a \geqslant 0$ and ω: $[0, a] \to X$ takes 0 and a to the basepoint of X. If $a > 0$ and f_a: $[0, a] \to I$ takes t to t/a, then $(a; \omega)$ determines a point $(a, \omega \circ f_a^{-1})$ of $\mathbf{R} \times \Omega X$; if $a = 0$ then ω is the trivial loop, and we associate $(0, \omega_0)$ to $(0; \omega)$, where $\omega_0 \in \Omega X$ is the constant loop of length 1. Under these identifications we topologize ΛX as a subspace of $\mathbf{R} \times \Omega X$.

[*Note*. The usual topology is given by making ω constant at the basepoint from a to ∞ and considering ΛX as a subspace of $\mathbf{R} \times X^{[0,\infty]}$. This agrees with the one we have defined, by an easy point-set-topological exercise.]

This device also simplifies the proofs above, with "a" playing the role of a variable height of the ambient n-cube. Let $\tilde{\mathcal{C}}'_{n,j}$ consist of $(j + 1)$-tuples $\langle a; c_1, \ldots, c_j\rangle$ with $a > 0$ and the c_r's linear embeddings, c_r: $[a_r^1, a_r^2] \times I^{n-1} \to [0, a] \times I^{n-1}$, satisfying (i) and (ii) above and the further condition that $m_r = \pm 1$ for all r. If $j = 0$ we allow $a \geqslant 0$.

Now for $a > 0$ the function f_a: $[0, a] \to I$ defined above induces a map

$$\mu' = \langle \mathrm{pr}_1, \mu\rangle: \tilde{\mathcal{C}}'_{n,j} \to R \times \tilde{\mathcal{C}}_{n,j}$$

given by $\mathrm{pr}_1\langle a; \mathbf{c}\rangle = a$ and $\mu\langle a; c_1, \ldots, c_j\rangle = \langle d_1, \ldots, d_j\rangle$, where

$$d_r = (f_a \times \mathrm{id}) \circ c_r.$$

We topologize $\tilde{\mathcal{C}}'_{n,j}$ as a subspace under this inclusion, and we wish to show further that μ is a homotopy equivalence.

In fact, we note that Σ_j acts in a natural way on $\tilde{\mathcal{C}}_{n,j}$ and on $\tilde{\mathcal{C}}'_{n,j}$ by permuting cubes, and we have the following lemma.

LEMMA 1.1. *The map* μ: $\tilde{\mathcal{C}}'_{n,j} \to \tilde{\mathcal{C}}_{n,j}$ *is a* Σ_j*-equivariant homotopy equivalence. If* $\partial\tilde{\mathcal{C}}_{n,j}$ *consists of configurations containing cubes of degenerate domain or with attached cubes of the same slope, then the inclusion* $\partial\tilde{\mathcal{C}}_{n,j} \to \tilde{\mathcal{C}}_{n,j}$ *is a cofibration, and a corresponding statement is true for* $\tilde{\mathcal{C}}'_{n,j}$.

The proof of this is left to §5.

We are now ready to define our approximating spaces. Let X be any based space. As noted in the lemma, Σ_j acts on $\tilde{\mathcal{C}}_{n,j}$ by permuting cubes. It also acts on the cartesian power X^j by permuting factors, and hence on the product $\tilde{\mathcal{C}}_{n,j} \times X^j$ diagonally. It is clear that this still defines an action when restricted to the subspace $\tilde{\mathcal{C}}_{n,j} \overline{\times} X^j$ of elements $(\langle c_1, \ldots, c_j\rangle, x_1, \ldots, x_j)$ for which $x_r = x_s$ whenever c_r and c_s are attached to each other. We define

$$\tilde{C}_nX = \Big(\coprod_{j \geqslant 0} \tilde{\mathcal{C}}_{n,j} \overline{\times} X^j/\Sigma_j\Big)/\sim$$

where the equivalence relation $\sim$ on the union of the Σ_j-orbit spaces is generated by the relations

$$(\langle c_1, \ldots, c_j\rangle, x_1, \ldots, x_j) \sim (\langle c_1, \ldots, c_{j-1}\rangle, x_1, \ldots, x_{j-1})$$

whenever x_j is the basepoint $*$ of X, or c_j is degenerate, and

$$(\langle c_1, \ldots, c_j\rangle, x_1, \ldots, x_j) \sim (\langle c_1, \ldots, c_{j-1} \cup c_j\rangle, x_1, \ldots, x_{j-1})$$

if c_{j-1} and c_j are attached and are both positive or negative; of course this implies that $x_{j-1} = x_j$. The $\sim$-equivalence classes are referred to by the notation $[\langle c_1, \ldots, c_j\rangle, x_1, \ldots, x_j]$, or $[\langle \mathbf{c}\rangle, \mathbf{x}]$ for short.

It is obvious how to define $\tilde{C}_n(f)$ for a continuous map $f\colon X_1 \to X_2$. One can therefore check that $\tilde{C}_n$ is a continuous covariant functor.

How does this new functor relate back to the theory of operads and specifically to $\mathcal{C}_n$-spaces? We answer this by giving an action of $\mathcal{C}_n$ on $\tilde{C}_nX$.

The action of $\mathcal{C}_n$ on $\tilde{C}_nX$ is given by composition of partial little cubes in $\tilde{C}_nX$ with little cubes in $\mathcal{C}_{n,k}$. Specifically, define

$$\gamma\colon \mathcal{C}_{n,k} \times \tilde{\mathcal{C}}_{n,j_1} \times \cdots \times \tilde{\mathcal{C}}_{n,j_k} \to \tilde{\mathcal{C}}_{n,j_1+\cdots+j_k}$$

by

$$\gamma(\langle b_1, \ldots, b_k\rangle; \langle c_{1,1}, \ldots, c_{1,j_1}\rangle, \ldots, \langle c_{k,1}, \ldots, c_{k,j_k}\rangle)$$
$$= \langle b_1 \circ c_1, \ldots, b_1 \circ c_{1,j_1}, \ldots, b_k \circ c_{k,1}, \ldots, b_k \circ c_{k,j_k}\rangle,$$

and then let the $\mathcal{C}_n$-structure map $\theta_k\colon \mathcal{C}_{n,k} \times (\tilde{C}_nX)^k \to \tilde{C}_nX$ be given by

$$\theta_k(\mathbf{b}; [\langle \mathbf{c}_1\rangle, \mathbf{x}_1], \ldots, [\langle \mathbf{c}_k\rangle, \mathbf{x}_k]) = [\langle \gamma(\mathbf{b}; \mathbf{c}_1, \ldots, \mathbf{c}_k)\rangle, \mathbf{x}_1, \ldots, \mathbf{x}_k].$$

This structure yields an H-space structure on $\tilde{C}_nX$ by use of a fixed choice of $\mathbf{b} \in \mathcal{C}_{n,2}$. Specifically, we choose $\mathbf{b} = \langle b_1' \times b'', b_2' \times b''\rangle$ where b'' is the identity on I^{n-1} and

$$b_i'(t) = \tfrac{1}{2}(i - 1 + t) \quad \text{for } i = 1 \text{ and } 2;$$

thus

$$\varphi([\langle \mathbf{c}\rangle, \mathbf{x}], [\langle \mathbf{d}\rangle, \mathbf{y}]) = [\langle \gamma(\mathbf{b}; \mathbf{c}, \mathbf{d})\rangle, \mathbf{x}, \mathbf{y}].$$

Similarly, $\tilde{C}_n'X$ is an associative topological monoid under the multiplication φ' defined by

$$\varphi'([\langle a; c_1, \ldots, c_j\rangle, x_1, \ldots, x_j], [\langle b; d_1, \ldots, d_k\rangle, y_1, \ldots, y_k])$$
$$= [\langle a + b; c_1, \ldots, c_j, T_a d_1, \ldots, T_a d_k\rangle, x_1, \ldots, x_j, y_1, \ldots, y_k]$$

where $T_a d_r$ is obtained from d_r by translation by a in the first coordinate. We have the following lemma:

LEMMA 1.2. *If X is a nondegenerately based space, then $\tilde{C}_nX$ and $\tilde{C}_n'X$ are homotopy-equivalent as H-spaces via the map μ induced from $\mu\colon \tilde{\mathcal{C}}_{n,j}' \to \tilde{\mathcal{C}}_{n,j}$.*

PROOF. First filter both spaces by the number of cubes; that is, let $F_j\tilde{C}_nX$ be the image of $\coprod_{r=0}^{j} \tilde{\mathcal{C}}_{n,r} \overline{\times} X^r$ and similarly for $F_j\tilde{C}_n'X$. Then it is clear that $F_j\tilde{C}_nX$ is built up from $F_{j-1}\tilde{C}_nX$ by the pushout diagram

$$\begin{array}{ccc} \left(\partial\tilde{\mathcal{C}}_{n,j} \,\overline{\times}\, X^j \cup \tilde{\mathcal{C}}_{n,j} \,\overline{\times}\, \bigvee X^j\right)/\Sigma_j & \rightarrow & F_{j-1}\tilde{C}_nX \\ \downarrow & & \downarrow \\ \tilde{\mathcal{C}}_{n,j} \,\overline{\times}\, X^j/\Sigma_j & \rightarrow & F_j\tilde{C}_nX \end{array}$$

where the "fat wedge" $\bigvee X^j$ consists of all j-tuples of points of X, at least one of which equals the basepoint, and the horizontal arrows are the identification maps induced by $\sim$. Then since X is nondegenerately based, $\bigvee X^j \hookrightarrow X^j$ is a Σ_j-equivariant cofibration, in the sense that it has a representation (k, v) as a strong NDR-pair where v: $X^j \to [0, \infty)$ is a Σ_j-invariant function and k_t: $X^j \to X^j$ is a Σ_j-equivariant homotopy. Further, the proof of Lemma 1 shows that $\partial\tilde{\mathcal{C}}_{n,j} \hookrightarrow \tilde{\mathcal{C}}_{n,j}$ is a Σ_j-equivariant cofibration, and that the homotopy h_t: $\tilde{\mathcal{C}}_{n,j} \to \tilde{\mathcal{C}}_{n,j}$ constructed there has the property that two cubes never become attached under h_t unless they were so at $t = 0$. Thus we can combine these representations to get a Σ_j-equivariant representation of $(\tilde{\mathcal{C}}_{n,j} \,\overline{\times}\, X^j, \partial\tilde{\mathcal{C}}_{n,j} \,\overline{\times}\, X^j \cup \tilde{\mathcal{C}}_{n,j} \,\overline{\times}\, \bigvee X^j)$ as an NDR-pair, and hence the left-hand inclusion in the diagram is a cofibration. The corresponding statements hold for $\tilde{\mathcal{C}}'_{n,j}$ as well.

The argument of [**1**, Theorem 2.7(ii)] is now easily adjusted to the current situation, and the first statement in Lemma 1 yields that μ: $\tilde{C}'_nX \to \tilde{C}_nX$ is a homotopy equivalence. The following diagram is homotopy-commutative:

$$\begin{array}{ccc} \tilde{C}'_nX \times \tilde{C}'_nX & \xrightarrow{\varphi'} & \tilde{C}'_nX \\ \mu \times \mu\downarrow & & \downarrow\mu \\ \tilde{C}_nX \times \tilde{C}_nX & \xrightarrow{\varphi} & \tilde{C}_nX \end{array}$$

and the lemma follows.

We need the following lemma, which helps us to believe that $\tilde{C}_nX$ actually approximates $\Omega^n\Sigma^nX$, and will be used at a crucial step later on.

LEMMA 1.3. *Under the given multiplications, $\tilde{C}_nX$ and $\tilde{C}'_nX$ are H-groups.*

PROOF. It clearly suffices to prove this for just C_nX, and we will give a based involution η: $\tilde{C}_nX \to \tilde{C}_nX$ such that $\varphi(1, \eta) \simeq 0$, whence it follows that $\varphi(\eta, 1) \simeq 0$ as well.

Let $\mathbf{c} = \langle c_1, \ldots, c_j\rangle$ be in $\tilde{\mathcal{C}}_{n,j}$, and let $\mathbf{d}$ be the result of "turning $\mathbf{c}$ upside down". That is, if $c_r = c'_r \times c''_r$, the corresponding d_r should have the same domain as c_r, and should satisfy $d_r(s, t) = (1 - c'_r(s), c''_r(t))$ there.

Let $\eta[\langle \mathbf{c}\rangle, \mathbf{x}] = [\langle \mathbf{d}\rangle, \mathbf{x}]$.

Now for $z \in \tilde{C}_nX$, $\varphi(z, \eta(z))$ consists of equal and opposite configurations facing each other across the "equator" $\frac{1}{2} \times I^{n-1}$ of I^n. The required homotopy is defined by moving both configurations toward each other at constant speed, attaching cubes when they meet at the equator, and dropping them from the configurations when they become degenerate. This can be made more precise, but less clear. □

2. The maps α_n, β_n, and β_n'. Throughout this section, let

$$z = [\langle c_1, \ldots, c_j\rangle, x_1, \ldots, x_j] \in \tilde{C}_n X.$$

Take $S^n = I^n/\partial I^n$. Thus for a based space X,

$\Sigma^n X = X \times I^n / * \times I^n \cup X \times \partial I^n$ and $\Omega^n X = \{\omega \in \mathrm{Map}(I^n, X) | \omega(\partial I^n) = *\}$. Using these notations, we define a natural map α_n: $\tilde{C}_n X \to \Omega^n \Sigma^n X$. With z as above, let $\alpha_n(z)$: $I^n \to \Sigma^n X$ be given by

$$\alpha_n(z)(u) = \begin{cases} [x_r, c_r^{-1}(u)] & \text{if } u \in \operatorname{im} c_r, \\ * & \text{if } u \notin \bigcup_r \operatorname{im} c_r \end{cases}$$

on elements $u \in I^n$.

To see that this is a continuous map, one notes that it is so on the image of each little cube, and is constant at the basepoint on the closure of $I^n - \bigcup_r \operatorname{im} c_r$. Hence it is continuous if it is well defined. The only possible difficulty occurs when $u \in \operatorname{im} c_r \cap \operatorname{im} c_s$ for two different indices r and s. Then

$$u = (c_r'(a_r^\varepsilon), c_r''(v)) = (c_s'(a_s^{-\varepsilon}), c_s''(w)),$$

and by the definition of $\tilde{\mathcal{C}}_{n,j}$ it follows that either $c_r^{-1}(u)$ and $c_s^{-1}(u)$ are on ∂I^n, so that $[x_r, c_r^{-1}(u)] = * = [x_s, c_s^{-1}(u)]$, or $0 < a_s^{-\varepsilon} < 1$, in which case $a_r^\varepsilon = a_s^{-\varepsilon}$ and $c_r'' = c_s''$. Then $c_r^{-1}(u) = (a_r^\varepsilon, v) = (a_s^{-\varepsilon}, w) = c_s^{-1}(u)$ and $x_r = x_s$. This shows that $\alpha_n(z)$ is well defined.

To see that α_n is itself continuous, consider the maps $\tilde{\alpha}_{n,j}$: $\tilde{\mathcal{C}}_{n,j} \mathbin{\overline{\times}} X^j \to \Omega^n\Sigma^n X$ induced by composing α_n with the identification map. The continuity of α_n will follow from that of each of the $\tilde{\alpha}_{n,j}$'s.

We wish to divide $\tilde{\mathcal{C}}_{n,j} \mathbin{\overline{\times}} X^j$ into small enough pieces that the continuity of $\tilde{\alpha}_{n,j}$ is visible on each. Accordingly, if R is any symmetric binary relation on the indices $1, \ldots, j$, we define the subspace $\tilde{\mathcal{C}}_{n,j}^R$ of $\tilde{\mathcal{C}}_{n,j}$ to consist of j-tuples $\langle c_1, \ldots, c_j\rangle$ such that c_r and c_s are adjacent if $r\, R\, s$, and are nonattached otherwise. For any R, this is a closed (possibly empty) subspace, and such subspaces cover $\tilde{\mathcal{C}}_{n,j}$ as R ranges over all symmetric binary relations. Thus to test the continuity of $\tilde{\alpha}_{n,j}$ we can restrict our attention to a single subspace $\tilde{\mathcal{C}}_{n,j}^R \mathbin{\overline{\times}} X^j$.

If $\langle c_1, \ldots, c_j\rangle \in \tilde{\mathcal{C}}_{n,j}^R$ and $\hat{R}$ is the equivalence relation generated by R, then we say that c_r and c_s are in the same *closed configuration* if $r\, \hat{R}\, s$. For example, if R is the empty relation, $\tilde{\mathcal{C}}_{n,j}^R$ consists of tuples of j little cubes, no two of which are attached, and the closed configurations in points of $\tilde{\mathcal{C}}_{n,j}^R$ consist of single cubes.

If $1 \leqslant k \leqslant j$ and i_k is the number of closed configurations containing exactly k cubes (equivalence classes under $\hat{R}$ of size k), then the operation of dropping all other closed configurations induces a map $\tilde{\mathcal{C}}_{n,j}^R \to (\tilde{\mathcal{C}}_{n,k}^{(k)})^{i_k}$, where $\tilde{\mathcal{C}}_{n,k}^{(k)}$ is the subspace of $\tilde{\mathcal{C}}_{n,k}$ of tuples of k cubes all in the same closed configuration. Letting k vary and extending to the product, we obtain an embedding

$$\tilde{\mathcal{C}}_{n,j}^R \mathbin{\overline{\times}} X^j \to \mathop{\times}_{k=1}^{j} \left(\tilde{\mathcal{C}}_{n,k}^{(k)} \mathbin{\overline{\times}} \Delta X^k\right)^{i_k}$$

where ΔX^k is the diagonal in X^k, and $j = \sum k i_k$. But $\tilde{\alpha}_{n,k}$ is easily seen to be continuous when restricted to $\tilde{\mathcal{C}}_{n,k}^{(k)} \mathbin{\overline{\times}} \Delta X^k$, and since we can fit the resulting compositions back together, the continuity of $\tilde{\alpha}_{n,j}$ on all $\tilde{\mathcal{C}}^R_{n,j} \overline{\times} X^j$ follows. Precisely, let $q = \Sigma^j_{k=1} i_k$, and consider the product

$$\mathop{\times}_{k=1}^{j} (\tilde{\alpha}_{n,k})^{i_k}: \mathop{\times}_{k=1}^{j} \left(\tilde{\mathcal{C}}^{(k)}_{n,k} \overline{\times} \Delta X^k\right)^{i_k} \to (\Omega^n \Sigma^n X)^q.$$

It is clear that this is continuous and takes $\tilde{\mathcal{C}}^R_{n,j} \overline{\times} X^j$ into the subspace $\Omega^n(\bigvee^q_{r=1} \Sigma^n X)$ under our identification. Restricting the domain and range to the latter two subspaces gives a continuous map $\bar{\alpha}^R_{n,j}$. Now composition with the q-fold folding map ∇: $\bigvee^q_{r=1} \Sigma^n X \to \Sigma^n X$ induces a map ∇_*: $\Omega^n(\bigvee^q_{r=1} \Sigma^n X) \to \Omega^n \Sigma^n X$, and $\tilde{\alpha}_{n,j}$ is the composite

$$\tilde{\mathcal{C}}^R_{n,j} \overline{\times} X^j \xrightarrow{\bar{\alpha}^R_{n,j}} \Omega^n\left(\bigvee_{r=1}^{q} \Sigma^n X\right) \xrightarrow{\nabla_*} \Omega^n \Sigma^n X.$$

Hence $\tilde{\alpha}_{n,j}$ is continuous, and hence so is α_n.

We pause to make a remark. The above discussion indicates that we might be able to filter $\tilde{C}_n X$ by the number of closed configurations, rather than by the number of cubes, as we have done. Indeed this is the case: if $j < k$, let $\tilde{\mathcal{C}}_{n,j,k}$ denote the subspace of $\tilde{\mathcal{C}}_{n,k}$ of tuples $\langle c_1, \ldots, c_k \rangle$ which can be joined into $\leqslant j$ closed configurations, and define $F'_j \tilde{C}_n X$ to be the image of

$$\coprod_{k=j}^{\infty} \tilde{\mathcal{C}}_{n,j,k} \overline{\times} X^k \approx \coprod_{k=j}^{\infty} \tilde{\mathcal{C}}_{n,j,k} \times X^j.$$

Then it is not hard to check that the inclusion $F'_{j-1} \tilde{C}_n X \to F'_j \tilde{C}_n X$ is a cofibration.

It is clear that the $\mathcal{C}_n$-structure given above for $\tilde{C}_n X$ agrees with the standard structure on $\Omega^n \Sigma^n X$ under α_n; thus the approximation theorem will show that $\tilde{C}_n X$ and $\Omega^n \Sigma^n X$ are weakly equivalent as $\mathcal{C}_n$-spaces.

For convenience we wish to factor α_n through $\Omega \alpha_{n-1}$, where the embedding α_{n-1}: $C_{n-1} \Sigma X \to \Omega^{n-1} \Sigma^n X$ is May's approximation [3]. If $z \in \tilde{C}_n X$ and $s \in I$, define b_s: $S^{n-1} \to \Sigma^n X$ by letting $b_s(t) = \alpha_n(z)(s, t)$ for $t \in I^{n-1}$. We will show that this map has image lying in that of α_{n-1}; it follows at once that α_n itself has image lying in Ω im α_{n-1} = im $\Omega \alpha_{n-1}$, and hence that $\alpha_n = \Omega \alpha_{n-1} \circ \beta_n$ for some continuous map β_n: $\tilde{C}_n X \to \Omega C_{n-1} \Sigma X$.

So choose $s \in I$, and examine $b_s(t) = \alpha_n(z)(s, t)$. If $s \notin \bigcup_r$ im c'_r, then b_s is the constant map at $*$, which is certainly in the image of α_{n-1}. Otherwise let $r_1, \ldots, r_k$ be the indices r for which $s = c'_r(s_r)$ for some $s_r \in I$, allowing at most one index from any attached pair. This latter clause is to ensure that the cubes c''_r for $r = r_1, \ldots, r_k$ are distinct even when two cubes are attached at s_r. Now regardless of the choices made, the value of $b_s(t)$ is $*$ if $t \notin \bigcup^k_{i=1} \operatorname{im}(c''_{r_i})$, and is $[x_r, s_r, u]$ if $t = c''_r(u)$ for $u \in I^{n-1}$ and r = one of $r_1, \ldots, r_k$. This is identical to the definition of $\alpha_{n-1}(y)(t)$, where

$$y = \left[\langle c''_{r_1}, \ldots, c''_{r_k} \rangle, [x_{r_1}, s_{r_1}], \ldots, [x_{r_k}, s_{r_k}]\right].$$

Therefore, $b_s \in$ im α_{n-1}, and this proves that $\alpha_n(z) \in$ im $\Omega \alpha_{n-1}$, by the above remark.

Finally, we need β_n': $\tilde{C}_n'X \to \Lambda C_{n-1}\Sigma X$. Let $z' = [\langle a; c_1, \ldots, c_j\rangle, x_1, \ldots, x_j]$. Then let $\beta_n'(z') = (a; \omega)$, where ω: $[0, a] \to C_{n-1}\Sigma X$ is given by

$$\omega(t) = \beta_n(\mu(z'))(t/a) \quad \text{if } a \neq 0.$$

We now have a commutative diagram:

$$\begin{tikzcd}
\tilde{C}_n'X \arrow[rr, "\beta_n'"] \arrow[d, "\mu", "\wr"'] & & \Lambda C_{n-1}\Sigma X \arrow[d, "\wr"] \\
\tilde{C}_nX \arrow[rr, "\beta_n"] \arrow[dr, "\alpha_n"'] & & \Omega C_{n-1}\Sigma X \arrow[dl, "\simeq"', "\Omega\alpha_{n-1}"] \\
& \Omega^n\Sigma^n X &
\end{tikzcd}$$

We will proceed to prove

THEOREM 2.1. *If X is a space with nondegenerate basepoint, then β_n' is a weak homotopy equivalence. Therefore β_n is a weak equivalence, and α_n: $\tilde{C}_nX \to \Omega^n\Sigma^n X$ is a weak equivalence of $\mathcal{C}_n$-spaces.*

3. The total space and the main diagram. Let ΠX denote the Moore paths in X, that is, pairs $(a; \omega)$ with $a \geqslant 0$ and ω: $[0, a] \to X$ such that $\omega(0) = *$. Then the image of β_n' in $\Lambda C_{n-1}\Sigma X \subseteq \Pi C_{n-1}\Sigma X$ is the subspace of all loops of the form

$$\omega(t) = \left[\langle c_1'', \ldots, c_k''\rangle, [x_1, \sigma_1(t)], \ldots, [x_k, \sigma_k(t)]\right]$$

where each σ_r is a continuous function $[0, a] \to I$ such that

(i) σ_r is piecewise linear with finitely many pieces,

(ii) on each piece, σ_r is either constant at 0 or 1, or has slope ± 1,

(iii) $\sigma_r(0) = 0$ or 1,

(iv) $\sigma_r(a) = 0$ or 1.

We will let our model E_nX of the path space consist of all partial paths of such loops; equivalently, define E_nX as the space of all Moore paths $(a; \omega)$ of the above form which satisfy (i), (ii), and (iii).

It is easy to see that E_nX is contractible via the standard path-space contraction, and that β_n maps $\tilde{C}_n'X$ into E_nX homeomorphically as $p^{-1}(*)$, where p is the restriction to E_nX of the endpoint projection map

$$p_e\colon \Pi C_{n-1}\Sigma X \to C_{n-1}\Sigma X.$$

The following diagram commutes:

$$\begin{tikzcd}
\tilde{C}_n'X \arrow[r, "\beta_n'"] \arrow[d, "\beta_n'"] & \Lambda C_{n-1}\Sigma X \arrow[d, hook] \\
E_nX \arrow[r, hook] \arrow[d, "p"] & \Pi C_{n-1}\Sigma X \arrow[d, "p_e"] \\
C_{n-1}\Sigma X \arrow[r, equal] & C_{n-1}\Sigma X,
\end{tikzcd}$$

and the main theorem will follow once it is shown that p is a quasifibration.

4. The quasifibration property. Recall the Dold-Thom criterion for a quasifibration over a filtered base space [2]. A subset V of $C_{n-1}\Sigma X$ is said to be *distinguished* if p: $p^{-1}(V) \to V$ is a quasifibration. The criterion implies that $C_{n-1}\Sigma X$ is distinguished if:

(i) every open subset of $F_jC_{n-1}\Sigma X - F_{j-1}C_{n-1}\Sigma X$ is distinguished, and $F_0C_{n-1}\Sigma X$ is distinguished, and

(ii) there is a deformation h_t of a neighborhood U of $F_{j-1}C_{n-1}\Sigma X$ in $F_jC_{n-1}\Sigma X$, and a covering homotopy H_t: $p^{-1}(U) \to p^{-1}(U)$ such that:

(1) h_0 is the identity and $h_1(U) \subseteq F_{j-1}C_{n-1}\Sigma X$,

(2) $H_0 = \mathrm{id}$ and for all $t, pH_t = h_tp$,

(3) for all $z \in U$, the map H_1: $p^{-1}(z) \to p^{-1}(h_1z)$

is a homotopy equivalence.

Here we give $C_{n-1}\Sigma X$ the filtration of [3]; that is, $F_jC_{n-1}\Sigma X$ is defined to be the image of $\amalg_{0 \leqslant k \leqslant j} \mathcal{C}_{n-1,k} \times (\Sigma X)^k$ under the identification.

PROOF OF (i). $F_0C_{n-1}\Sigma X = *$, which is obviously distinguished.

Let V be an open set in $F_j - F_{j-1}$. We will construct maps

$$p^{-1}(V) \underset{w}{\overset{\langle p, q\rangle}{\rightleftarrows}} V \times \tilde{C}_n'X$$

which will be inverse homotopy equivalences over V; it follows that V is distinguished.

Let $y = [\langle c_1'', \dots, c_j''\rangle, [x_1, s_1], \dots, [x_j, s_j]] \in V$. We can assume $x_1, \dots, x_j \in X - *$ and $s_1, \dots, s_j \in (0, 1)$, so it makes sense to define $s = \max_r s_r$. Then we can let $w(y, 0)$ be the path with slopes $+1$ from the basepoint to y, where 0 denotes the element of height zero in $\tilde{C}_n'X$. That is, $w(y, 0) = (s; \pi)$ where

$$\pi(t) = \left[\langle \mathbf{c}''\rangle, \left[x_1, \sigma_1(t)\right], \dots, \left[x_j, \sigma_j(t)\right]\right], \qquad 0 \leqslant t \leqslant s,$$

with $\sigma_r(t) = \max(s_r - s + t, 0)$. Note that $\pi(s) = y$.

Now for any $z \in \tilde{C}_n'X$ define

$$w(y, z) = \beta_n(z) + w(y, 0)$$

where the + denotes addition of loops and paths.

We have already defined p, and it remains to define q: $p^{-1}V \to \tilde{C}_n'X$. Let $(a; \omega) \in p^{-1}V$, with $y = \omega(a)$ as above. Then let

$$q(a; \omega) = \beta_n^{-1}((a; \omega) + (-w(y, 0)))$$

where "$-w(y, 0)$" denotes the reverse of the path $w(y, 0)$.

The path $w(y, 0) + (-w(y, 0))$ is actually a canonically contractible loop at the basepoint, and $(-w(y, 0)) + w(y, 0)$ is a canonically contractible loop at y, and so it is obvious that $\langle p, q\rangle \circ w$ and $w \circ \langle p, q\rangle$ are homotopic to the respective identities over V.

PROOF OF (ii). Here we assume that $(X, *)$ is a strong NDR-pair (see the appendix to [3]); this implies that there is a neighborhood V of $*$ and a homotopy k_t: $X \to X$ sending V to itself such that k_0 is the identity and $k_1(V) = *$. Let

$$U = \left\{y \in F_jC_{n-1}X \,|\, y = \left[\langle \mathbf{c}''\rangle, \left[x_1, s_1\right], \dots, \left[x_j, s_j\right]\right]\right.$$
$$\left. \text{and for some } r, s_r < \tfrac{1}{3} \text{ or } s_r > \tfrac{2}{3} \text{ or } x \in V\right\}.$$

Consider the function $f\colon I \to I$ defined by

$$f(u) = \begin{cases} 0 & \text{if } u \leqslant \frac{1}{3}, \\ (3u-1) & \text{if } \frac{1}{3} \leqslant u \leqslant \frac{2}{3}, \\ 1 & \text{if } \frac{2}{3} \leqslant u. \end{cases}$$

Let

$$f_{3v}(u) = \begin{cases} 0 & \text{if } u \leqslant v, \\ u - v & \text{if } v \leqslant u \leqslant \frac{1}{3}, \\ 3u - v - \frac{2}{3} & \text{if } \frac{1}{3} \leqslant u \leqslant v + \frac{1}{3}, \\ u + v & \text{if } v + \frac{1}{3} \leqslant u \leqslant 1 - v, \\ 1 & \text{if } 1 - v \leqslant u \end{cases}$$

for $0 \leqslant v \leqslant \frac{1}{3}$. Then f_t induces a homotopy $\tilde{f}_t\colon S^1 \to S^1$, and recalling the definition $\Sigma X = X \wedge S^1$, we can define a deformation $h_t\colon U \to U$ by letting

$$h_t(y) = C_{n-1}(k_t \wedge \tilde{f}_t)(y) \quad \text{for } 0 \leqslant t \leqslant 1.$$

This can be covered by a homotopy H_t as follows: let $(a; \omega) \in p^{-1}(U)$ with $\omega(a) = y$. Then define

$$H_t(a; \omega) = (a + t/3; \bar{\omega}_t)$$

where

$$\bar{\omega}_t(u) = \begin{cases} C_{n-1}(k_t \wedge 1)(\omega(u)) & \text{if } u \leqslant a, \\ C_{n-1}(k_t \wedge \tilde{f}_{3(u-a)})(y) & \text{if } a \leqslant u \leqslant a + t/3. \end{cases}$$

Now clearly $f_{3(u-a)}$ is piecewise linear with slopes ± 1 for $0 \leqslant u - a \leqslant \frac{1}{3}$, and so $\bar{\omega}_t$ is a path in $E_n X$ ending at $h_t(y)$. Hence H_t covers h_t.

It is trivial to verify that $h_t(U) \subseteq U$, that h_0 and H_0 are the respective identities, and that $h_1(U) \subseteq F_{j-1}$. Hence we are left with showing that H_1 is an equivalence on fibers.

But suppose $y \in U$, and let ξ be right translation in $\tilde{C}'_n X$ by the element whose image under β_n is the loop

$$H_1(w(y, 0)) + (-w(h_1(y), 0)).$$

Then ξ is an equivalence by Lemma 1.3, and

$$\begin{array}{ccc} p^{-1}(y) & \xrightarrow{H_1} & p^{-1}(h_1(y)) \\ w\uparrow & & \downarrow\langle p, q\rangle \\ \{y\} \times \tilde{C}'_n X & \xrightarrow{h_1 \times \xi} & \{h_1(y)\} \times \tilde{C}'_n X \end{array}$$

commutes. But w and $\langle p, q\rangle$ are equivalences. Hence H_1 is an equivalence.

5. Proof of Lemma 1.1. Let $\bar{\mathcal{C}}_{n,j}$ be defined precisely as $\tilde{\mathcal{C}}'_{n,j}$ except that the slopes m_r are allowed to be any nonzero number, rather than just ± 1. Then the map μ factors as

$$\tilde{\mathcal{C}}'_{n,j} \xrightarrow{i} \bar{\mathcal{C}}_{n,j} \xrightarrow{\bar{\mu}} \tilde{\mathcal{C}}_{n,j}$$

where i is the obvious inclusion and

$$\bar{\mu}\langle a; c_1, \ldots, c_j\rangle = \langle (f_a \times \mathrm{id}) \circ c_1, \ldots, (f_a \times \mathrm{id}) \circ c_j\rangle.$$

Now with the condition on m_r relaxed we have an inclusion $\tilde{\mathcal{C}}_{nj} \to \bar{\mathcal{C}}_{nj}$ defined by sending $\langle \mathbf{c}\rangle$ to $\langle 1; \mathbf{c}\rangle$, and this inclusion is a Σ_j-equivariant homotopy inverse to $\bar{\mu}$ via the evident linear homotopies. Thus we are left with proving that $\tilde{\mathcal{C}}'_{nj}$ is a Σ_j-equivariant-deformation retract of $\bar{\mathcal{C}}_{n,j}$.

The deformation is constructed in two stages. For the first stage we recall the language of the definition of α_n. The domain of any cube c_r is of the form $[a_r^1, a_r^2] \times I^{n-1}$, and if $\mathcal{K}$ is a closed configuration we will define the *collective domain* of $\mathcal{K}$ to be the set $\mathfrak{D}(\mathcal{K}) = \bigcup_{c_r \in \mathcal{K}} [a_r^1, a_r^2]$, that is, the union of the first coordinates of the domains of cubes in $\mathcal{K}$.

By the definition of $\bar{\mathcal{C}}_{n,j}$, if $\langle a; \mathbf{c}\rangle \in \bar{\mathcal{C}}_{n,j}$ then $\mathfrak{D}(\mathcal{K})$ will either be all of $[0, 1]$ or will be contained in $[0, 1)$ or $(0, 1]$.

Our first goal will be to deform $\bar{\mathcal{C}}_{nj}$ inside the space $\bar{\mathcal{C}}_{nj}^{(S)}$ of points $\langle a; \mathbf{c}\rangle$ satisfying the following condition:

(S) If c_r belongs to a configuration $\mathcal{K}$ with $\mathfrak{D}(\mathcal{K}) = [0, 1]$, then $|m_r| > 1$.

Obviously $\tilde{\mathcal{C}}'_{n,j} \subseteq \bar{\mathcal{C}}_{n,j}^{(S)}$.

It seems apparent that if a cube $\langle a; \mathbf{c}\rangle$ is "stretched out" far enough it will get inside $\bar{\mathcal{C}}_{n,j}^{(S)}$; the present problem is to define "far enough" in a continuous manner, which we do as follows. Let

$$m' = \sum_{r=1}^{j} 3 \operatorname{meas}\left(\operatorname{dom}(c'_r) \cap \left[\frac{1}{3}, \frac{2}{3}\right]\right) \cdot \frac{1}{|m_r|}$$

where meas(J) assigns to any interval J its measure, and $m = \max(m', 1)$. Then we define $\lambda_t\colon \bar{\mathcal{C}}_{n,j} \to \bar{\mathcal{C}}_{n,j}$ by

$$\lambda_t\langle a; \mathbf{c}\rangle = \langle (1 - t + tm)a; \mathbf{d}\rangle,$$

where

$$d_r = f^{-1}_{(1-t+tm)a} \circ f_a \circ c_r\colon \left[a_r^1, a_r^2\right] \times I^{n-1} \to \left[0, (1 - t + tm)\right] \times I^{n-1}.$$

This is obviously the identity for $t = 0$, and at $t = 1$ the cubes d_r have slope $m \cdot m_r$. To see that λ_1 has image in $\bar{\mathcal{C}}_{n,j}^{(S)}$, suppose $d_r \in \mathcal{K}$, where $\mathcal{K}$ is a closed configuration in $\langle ma; \mathbf{d}\rangle = \lambda_1\langle a; \mathbf{c}\rangle$. Then if $\mathfrak{D}(\mathcal{K}) = [0, 1]$, the corresponding configuration $\mathcal{K}'$ in $\mathbf{c}$ must have had $\mathfrak{D}(\mathcal{K}') = [0, 1]$. Choose cubes $c_{r_1}, \ldots, c_{r_k}$ in $\mathcal{K}'$ with

$$\bigcup_{i=1}^{k} \operatorname{dom} c'_{r_i} \supseteq \left[\frac{1}{3}, \frac{2}{3}\right];$$

then

$$m \geq \sum_{i=1}^{k} 3 \operatorname{meas}\left(\operatorname{dom} c'_{r_i} \cap \left[\frac{1}{3}, \frac{2}{3}\right]\right) \cdot \frac{1}{|m_{r_i}|} > \frac{1}{|m_r|}.$$

Thus the slope of d has absolute value $|m \cdot m_r| \geq 1$.

The continuity of λ_t follows from that of m, and this holds because

$$\operatorname{meas}\left(\operatorname{dom}(c_r') \cap \left[\frac{1}{3}, \frac{2}{3}\right]\right)$$

is a continuous function of a_r^1 and a_r^2, and because the factor $1/|m_r|$ is kept from going out of control by the fact that $[\frac{1}{3}, \frac{2}{3}]$ is bounded away from the endpoints of I. Hence λ_t deforms $\bar{\mathcal{C}}_{n,j}$ into $\bar{\mathcal{C}}_{n,j}^{(S)}$.

One may object that λ_t is not the identity on $\bar{\mathcal{C}}_{n,j}^{(S)}$ for all t, but at least $\lambda_t(\bar{\mathcal{C}}_{n,j}^{(S)}) \subseteq \bar{\mathcal{C}}_{n,j}^{(S)}$, so the inclusion of $\bar{\mathcal{C}}_{n,j}^{(S)}$ in $\bar{\mathcal{C}}_{n,j}$ is a Σ_j-equivariant homotopy equivalence, which is all we really need.

Now we give a retraction ν of $\bar{\mathcal{C}}_{n,j}^{(S)}$ onto $\tilde{\mathcal{C}}_{n,j}'$. Whereas λ made adjustments in the "vertical" direction, ν will adjust the "horizontal" direction by affecting the domains of the c_r's. For example, if $\mathcal{K}$ is a closed configuration with collective domain in $[0, 1)$ or $(0, 1]$ and containing cubes whose slopes are in the interval $(-1, 0) \cup (0, 1)$, it is easy to shrink down the domains of the cubes coherently, keeping the images the same, so as to obtain cubes with slopes ± 1. The fact that our tuples satisfy (S) will let us perform a similar operation on all closed configurations.

Recall the notations a_r^- and a_r^+ from the first definition. Then if $\langle a; \mathbf{c}\rangle \in \bar{\mathcal{C}}_{n,j}^{(S)}$, recursively define pairs of numbers (b_r, e_r) as follows:

$$b_r = \begin{cases} a_r^- & \text{if } a_r^- = 0 \text{ or } 1, \\ e_s & \text{if } c_r \text{ is attached from below by } c_s, \end{cases}$$

and

$$e_r = \min\left(\max\left(0, b_r + c_r'(a_r^2) - c_r'(a_r^1)\right), 1\right),$$

so e_r is just $b_r + c_r'(a_r^2) - c_r'(a_r^1)$ adjusted to lie within I.

These pairs give us the domains for the cubes in $\nu\langle a; \mathbf{c}\rangle$. Namely, define d_r': $[\min(b_r, e_r), \max(b_r, e_r)] \to I$ by requiring it to be linear and defining

$$d_r'(b_r) = c_r'(a_r^-) \quad \text{and} \quad d_r'(e_r) = d_r'(b_r) + |e_r - b_r|.$$

Then let $d_r = d_r' \times c_r''$ for all r, and define $\nu\langle a; \mathbf{c}\rangle = \langle a; \mathbf{d}\rangle$.

We must check that $\langle a; \mathbf{d}\rangle$ is a well-defined element of $\tilde{\mathcal{C}}_{n,j}'$. First note that since $|e_r - b_r| \leqslant |c_r'(a_r^2) - c_r'(a_r^1)| = c_r'(a_r^+) - c_r'(a_r^-)$, the image of d_r is contained in that of c_r, so the images of $d_1, \ldots, d_j$ will still be disjoint on interiors. Obviously each d_r' is a linear map with slope ± 1, so it only remains to check the attachment condition (ii). For that we need a technical lemma giving us control over the b_r's and e_r's.

LEMMA 5.1. *Let $\mathcal{K}$ be a closed configuration in $\langle a; \mathbf{c}\rangle \in \bar{\mathcal{C}}_{n,j}^{(S)}$. Then if $\mathfrak{D}(\mathcal{K}) = [0, 1]$, $1 - |m_r| + |m_r| \cdot a_r^- \leqslant b_r \leqslant |m_r| a_r^-$ for all $c_r \in \mathcal{K}$. Otherwise either $\mathfrak{D}(\mathcal{K}) \subseteq [0, 1)$ and $b_r = |m_r| a_r^-$, or $\mathfrak{D}(\mathcal{K}) \subseteq (0, 1]$ and $b_r = 1 - |m_r| + |m_r| \cdot a_r^-$ for all $c_r \in \mathcal{K}$.*

PROOF. This is proved by induction on the number of cubes "below" c_r in $\mathcal{K}$. That is, we show as the base of the induction that the appropriate relation holds if c_r is not attached from below, and then in the induction step we suppose that c_r is attached from below by c_s and then show that the same relations hold for c_r as for c_s.

So suppose that $c_r \in \mathcal{K}$ is not attached from below. Then $b_r = a_r^- = 0$ or 1. If $\mathfrak{D}(\mathcal{K}) = [0, 1]$ then by (S), $|m_r| \geqslant 1$ so that

$$b_r \leqslant |m_r|a_r^- \quad \text{and} \quad (1 - b_r) \leqslant |m_r|(1 - a_r^-).$$

If $\mathfrak{D}(\mathcal{K}) = [0, 1)$ then $b_r = a_r^- = 0$ and so $b_r = |m_r|a_r^-$, and similarly if $\mathfrak{D}(\mathcal{K}) = (0, 1]$, $b_r = 1 - |m_r| + |m_r|a_r^-$.

Now suppose c_r is attached from below by c_s. Then $b_r = e_s$. If this is 0 or 1, then the above argument shows that the appropriate relation holds. Otherwise $e_s \in (0, 1)$ and so by definition

$$\begin{aligned} e_s &= b_s + c_s'(a_s^2) - c_s'(a_s^1) = b_s + m_s \cdot (a_s^2 - a_s^1) \\ &= b_s + |m_s|(a_s^+ - a_s^-), \end{aligned}$$

and since $|m_r| = |m_s|$ and $a_s^+ = a_r^-$, we obtain

$$b_s - |m_s|a_s^- = b_r - |m_r|a_r^-.$$

Hence b_r and $|m_r|a_r^-$ satisfy the same relation as b_s and $|m_s|a_s^-$, and similarly for b_r and $1 + |m_r| - |m_r|a_r^-$. $\square$

This lemma implies that condition (ii) holds for $\langle a; \mathbf{d}\rangle$. To see this, suppose that $0 < b_r < 1$. Then if $\mathfrak{D}(\mathcal{K}) = [0, 1]$, $|m_r|a_r^- > 0$ and $|m_r|(1 - a_r^-) > 0$; if $\mathfrak{D}(\mathcal{K}) \subseteq [0, 1)$ then $0 < b_r/|m_r| = a_r^- < 1$, and if $\mathfrak{D}(\mathcal{K}) \subseteq (0, 1]$ then $0 < (1 - b_r)/|m_r| = 1 - a_r^- < 1$. In any case $0 < a_r^- < 1$ and c_r is attached from below by some cube c_s. But then we have $b_r = e_s \in (0, 1)$, $|e_s - b_s| = c_s'(a_s^+) - c_s'(a_s^-)$, and so

$$\begin{aligned} d_s'(e_s) &= c_s'(a_s^-) + c_s'(a_s^+) - c_s'(a_s^-) \\ &= c_s'(a_s^+) = c_r'(a_r^-) = d_r'(b_r). \end{aligned}$$

Hence d_r is attached from below by d_s. On the other hand, if $0 < e_r < 1$, then a similar argument shows that $0 < a_r^+ < 1$; hence c_r is attached from below by some cube c_s. Then we compute that $b_s = e_r$, $d_s'(b_s) = d_r'(e_r)$. In each $|m_r| = |m_s| = 1$, so all the conditions for (ii) are satisfied.

We have demonstrated that $\nu\colon \bar{\mathcal{C}}_{n,j}^{(S)} \to \tilde{\mathcal{C}}'_{n,j}$ is well defined, and we must check that it is continuous. By the argument given for α_n, it will be enough to show that ν is continuous on a single closed configuration. So suppose $\langle a; \mathbf{c}\rangle$ consists of a single closed configuration, and suppose further that all the cubes in $\mathbf{c}$ are actually attached. Then a neighborhood about $\langle a; \mathbf{c}\rangle$ will still have all cubes attached and the numbers b_r, e_r for any point in this neighborhood will be calculated by the same formulae, and one can see that they vary continuously with $\langle a; \mathbf{c}\rangle$. Now if some pair of cubes, c_r and c_s, is adjacent but not attached, with $a_s^+ = a_r^- = 0$ or 1, then the argument given to check condition (ii) implies that $e_s = a_s^+$ so that $b_r = a_r^- = a_s^+ = e_s$. Thus in a neighborhood of $\langle a; \mathbf{c}\rangle$ the formulae for b_r and e_r still hold and hence vary continuously. It follows that ν is continuous.

Now one can see that instead of adjusting all the slopes m_r to ± 1, we could use the above procedure to adjust the slopes to any fixed fraction u of the difference between $|m_r|$ and 1, replacing m_r by $(1 - u)m_r + u \cdot \operatorname{sgn}(m_r)$. As u varies from 0 to

1, this defines a homotopy from the identity to ν, which is constantly the identity on $\tilde{\mathcal{C}}'_{n,j}$. Hence $\tilde{\mathcal{C}}'_{n,j}$ is a deformation retract of $\tilde{\mathcal{C}}^{(S)}_{n,j}$, and the symmetry inherent throughout this discussion implies that the map μ is a Σ_j-equivariant homotopy equivalence. $\square$

We now prove the second statement of Lemma 1 for $\tilde{\mathcal{C}}_{n,j}$; the proof goes over word-for-word to a proof of the corresponding statement for $\tilde{\mathcal{C}}'_{n,j}$.

Let $\partial'\tilde{\mathcal{C}}_{n,j}$ denote the subspace of configurations containing attached cubes with equal slopes. Then define u: $\tilde{\mathcal{C}}_{n,j} \to [0, 1]$ by

$$u\langle c_1, \ldots, c_j\rangle = \begin{cases} 0 & \text{if } \langle c_1, \ldots, c_j\rangle \in \partial'\tilde{\mathcal{C}}_{n,j}, \\ \min\limits_{r=1,\ldots,j} \left(a_r^2 - a_r^1\right) & \text{otherwise.} \end{cases}$$

Degenerate cubes have $a_r^1 = a_r^2$ so $u^{-1}(0) = \partial\tilde{\mathcal{C}}_{n,j}$. Further the only cluster points of $\partial'\tilde{\mathcal{C}}_{n,j}$ outside $\partial'\tilde{\mathcal{C}}_{n,j}$ have degenerate cubes; hence u is continuous. This map will be half of the data required to show that $(\tilde{\mathcal{C}}_{n,j}, \partial\tilde{\mathcal{C}}_{n,j})$ is an NDR-pair, proving the second statement. The other data will be a homotopy h_t: $\tilde{\mathcal{C}}_{n,j} \to \tilde{\mathcal{C}}_{n,j}$ with h_0 the identity, $h_t|\partial\tilde{\mathcal{C}}_{n,j}$ the identity on $\partial\tilde{\mathcal{C}}_{n,j}$ for all t, and with h_1 retracting the neighborhood $u^{-1}[0, \frac{1}{3})$ into $\partial\tilde{\mathcal{C}}_{n,j}$.

This is defined as follows. First, if $\mathbf{c} = \langle c_1, \ldots, c_j\rangle$, define numbers ε_r, called the *excess* of c_r, by

$$\varepsilon_r = \begin{cases} 0 \quad \text{if } c_r \text{ is not attached from below,} \\ \text{the measure of } \left[a_s^1, a_s^2\right] \cap \left[a_r^1, a_r^2\right] \quad \text{if } c_r \text{ is attached by } c_s \text{ from below.} \end{cases}$$

The number ε_r represents the maximum extent to which c_r can be coalesced along its bottom face. These numbers do not vary continuously with $\mathbf{c}$, but in fact all the discontinuities occur where $a_r^- = 0$ or 1. Therefore let $g(t) = \min(3t, 1)$ for $t \in I$, and define

$$\varepsilon_r' = g\left(|(a_r^-) - \varepsilon|\right) \cdot \varepsilon_r$$

where $\varepsilon = 0$ if $m_r > 0$ and $\varepsilon = 1$ if $m_r < 0$. Then ε_r' clearly varies continuously with $\mathbf{c}$. For technical reasons we modify them again by letting $\bar{\varepsilon}_r = \min(\varepsilon_r', u(\mathbf{c}))$.

Fix $t \in I$. We can "shorten" each cube in $\mathbf{c}$ by an amount $t\varepsilon_r$ as follows. Let b_r and e_r be defined recursively by:

$$b_r = \begin{cases} a_r^- & \text{if } c_r \text{ is not attached from below,} \\ e_s & \text{if } c_r \text{ is attached from below by } c_s, \end{cases}$$

$$e_r = \begin{cases} a_r^+ & \text{if } c_r \text{ is not attached from above,} \\ \max(b_r, a_r^+ - t\bar{\varepsilon}_s) & \text{if } c_r \text{ is attached from above by } c_s \text{ and } a_r^- < a_r^+, \\ \min(b_r, a_r^+ + t\bar{\varepsilon}_s) & \text{if } c_r \text{ is attached from above by } c_s \text{ and } a_r^+ < a_r^-. \end{cases}$$

e_r is well defined because if $a_r^- = a_r^+$, $\mathbf{c} \in \partial\tilde{\mathcal{C}}_{n,j}$ and so $\varepsilon_s = 0$ for all s. In fact, the technical point referred to above guarantees that if $u(\mathbf{c}) = 0$ or $t = 0$, then $b_r = a_r^-$ and $e_r = a_r^+$ for all r.

Now define linear maps d_r': $[\min(b_r, e_r), \max(b_r, e_r)] \to I$ by specifying

$$d_r'(b_r) = \begin{cases} c_r'(a_r^-) & \text{if } a_r^- = 0 \text{ or } 1, \\ d_s'(e_s) & \text{if } c_r \text{ is attached by } c_s \text{ from below,} \end{cases}$$

and

$$d_r'(e_r) = d_r'(b_r) + m_r \cdot (e_r - b_r).$$

That this actually takes values in I follows from an exercise which shows that $c_r'(a_r^+) \geqslant d_r'(e_r)$ for all r. Similarly one checks that $\langle \mathbf{d}' \times \mathbf{c}'' \rangle = \langle d_1' \times c_1'', \ldots, d_j' \times c_j'' \rangle$ satisfies conditions (i) and (ii) for $\tilde{\mathcal{C}}_{n,j}$ and we can define $h_t(\mathbf{c}) = \langle \mathbf{d}' \times \mathbf{c}'' \rangle$. The continuity of h follows from that of the numbers $\bar{\varepsilon}_r$.

Now the facts noted about the numbers b_r and e_r translate to the statement that h_0 is the identity and that h_t is the identity on $u^{-1}(0)$ for all t. Thus we need only show that $h_1(u^{-1}[0, \frac{1}{3}))$ is contained in $\partial\tilde{\mathcal{C}}_{n,j}$.

If $0 < u(\mathbf{c}) < \frac{1}{3}$ then let c_r have minimal domain, so that $[a_r^1, a_r^2]$ is wholly contained in either $[0, \frac{2}{3}]$ or $[\frac{1}{3}, 1]$. We suppose $m_r > 0$; the argument for $m_r < 0$ is similar. If $[a_r^1, a_r^2] \subseteq [\frac{1}{3}, 1]$ then $g(|a_r^- - 0|) = 1$ and $\bar{\varepsilon}_r = \varepsilon_r = a_r^2 - a_r^1 = u(\mathbf{c})$. (Since $a_r^- = a_r^1 \in (0, 1)$, c_r is attached from below by a cube c_q with $[a_q^1, a_q^2] \cap [a_r^1, a_r^2] = [a_r^1, a_r^2]$.) Hence either $b_q = e_q$ so that d_q' is degenerate, or $e_q = a_q^+ - \bar{\varepsilon}_r = a_r^1 + (a_r^2 - a_r^1) = a_r^2$. Thus $b_r = a_r^2$, and $e_r = \max(b_r, a_r^2 - \bar{\varepsilon}_s) = b_r$, if c_r is attached from above by c_s, or $e_r = a_r^2 = b_r$ if not. Hence d_r' is degenerate.

On the other hand suppose $[a_r^1, a_r^2] \subseteq [0, \frac{2}{3}]$. Then $a_r^+ = a_r^2 \in (0, 1)$ so that c_r is attached from above by a cube c_s. Then $[a_r^1, a_r^2] \subseteq [a_s^1, a_s^2] \subseteq [0, \frac{2}{3}]$, the first inclusion following by the condition $a_r^2 = a_s^2$ and minimality, and the second by the fact that $a_s^1 \leqslant a_s^2 = a_r^2 \leqslant \frac{2}{3}$. Now it follows that $\bar{\varepsilon}_s = \varepsilon_s = a_r^2 - a_r^1$, and since $b_r \geqslant a_r^- = a_r^1$, $e_r = \max(b_r, a_r^+ - \bar{\varepsilon}_s) = \max(b_r, a_r^2 - (a_r^2 - a_r^1)) = b_r$. Thus d_r' is degenerate. It follows, then, that $h_1(\mathbf{c}) \in \partial\tilde{\mathcal{C}}_{n,j}$ in either case. This completes the proof. $\square$

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